

# Microwave “optical” conductance

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We measure the transmission matrix for microwave radiation propagating through ensembles of random waveguides of different lengths and scattering strength. We find that the probability distribution of the transmittance  $T$ , the analog of the electronic conductance, evolves from Gaussian to log-normal and passes through a one-sided log-normal distribution in the crossover to Anderson localization. This asymmetrical distribution obtained for an ensemble with a value of the dimensionless conductance  $g$  of 0.37 is explained within the framework of an intuitive Coulomb gas model for the eigenvalues of the transmission matrix.

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Mesoscopic fluctuations of conductance and transmission limit the reproducibility of integrated circuits and the possibilities of imaging through opaque samples [1, 2]. At the same time, however, fluctuations of the field within disordered samples may also be exploited to lower the threshold for lasing [3] and sharpen focusing in random systems [4, 5]. The importance of fluctuations in electronics was first recognized in conduction mediated by localized states [6–9], but their impact was soon observed in universal fluctuations of conductance in samples in which electrons freely diffuse [10, 11] and for classical waves, such as light, microwave radiation and ultrasound [12]. Such fluctuations make it impossible to predict the characteristics of an individual disordered sample and, conversely, transport can not be characterized by the conductance or transmission in a single sample or even by averages over many samples. However, an account of fluctuations via probability distributions of conductance in ensembles of random samples with different physical dimensions could provide a full description of transport [6–8]. The corresponding measurements have not been possible in electronic systems because of the difficulty of producing collections of statistically equivalent samples. Measurements of fluctuations of conductance are typically made in single physical samples by varying the applied voltage or magnetic field [10].

Though systematic measurements of the statistics of conductance over ensembles of random samples have not been made, extensive measurements of fluctuation of transmitted intensity and total transmission for a single incident channel [12] and of correlation between channels [13] have been carried out for classical waves. However, measurements of the distribution of the transmittance,  $T$ , which is the sum of total transmitted flux over all incident channels, have not yet been reported.  $T$  is equivalent to the conductance in units of the quantum of conductance and is often called the “optical” conductance since it is the sum of transmission coefficients that can in principle be measured optically [14]. The localization threshold lies at  $\langle T \rangle = g = 1$ , where  $\langle \dots \rangle$  indicates the ensemble average [15].

The fullest measurement of transmission is the field transmission matrix  $t$  whose elements  $t_{ba}$  relate the field in outgoing channels  $b$  to the field in incident channels  $a$ ,  $E_b = \sum_{n=1}^N t_{ba} E_a$  [16, 17]. Here  $N$  is the number of propagating channels supported by the sample leads at a given energy or frequency.  $T$  may be expressed in terms of the elements of  $t$  and also in terms of the eigenvalues  $\tau_n$  of the matrix product  $tt^\dagger$ ,  $T = \sum_{a,b=1}^N |t_{ba}|^2 = \sum_{n=1}^N \tau_n$ . Optical measurements of the transmission matrix have been exploited recently to focus light transmitted through strongly scattering media [5]. But the dimensions of the matrix were too small for the impact of mesoscopic correlation on transmission to be manifest.

Simulations [18] have found a highly asymmetrical distribution of the logarithm of conductance just beyond the localization threshold in agreement with random matrix theory calculations [19, 20] for quasi-1D samples. However, because of the challenges of electronic and optical measurements, distributions of the electronic or optical conductance have not been made and the origin of the anomalous distribution which emerge in simulations and calculations have not been further clarified.

In this Letter, we present measurements of the transmission matrix for microwave radiation propagation through ensembles of random waveguides with the values of  $g$  ranging from 6.9 to 0.04. In the limits of diffusive and localized waves, the probability distribution of the optical conductance,  $P(T)$ , is found to be a Gaussian and log-normal distribution, respectively. For the ensemble with value of  $g = 0.37$  just beyond the localization threshold, a highly asymmetric distribution of  $\ln T$  is observed, which is log-normal for  $T$  below the peak of the distribution and exhibits an exponential tail for high values of  $T$ . This anomalous distribution for  $T$  can be explained by the repulsion between charges and their images associated with the transmission eigenvalues in a model developed in ref. [17], based on Dyson’s Coulomb model for eigenvalues of large random Hamiltonian matrix [21]. For low values of  $g$ , distributions of  $T$  become log-normal and the variance of the  $\ln T$  approach  $-(\ln T)$  as predicted in the single parameter scaling theory (SPS)

of localization for one dimensional systems [6, 7].

The scattering media studied are collections of randomly positioned alumina spheres with a diameter of 0.95 cm and refractive index 3.14 embedded in Styrofoam shells [22]. Those samples are contained within a copper tube of diameter 7.3 cm giving an alumina sphere volume fraction of 0.068. The transmission matrix is obtained by measuring  $t_{ba}$  between arrays of points on the incident and output surface for microwave radiation polarized along the orientation of the wire antenna with use of a vector network analyzer. The dimensionality of the measured transmission matrix matches the number of propagating waveguide modes supported by the copper tube, with  $N \sim 66$  in the frequency range 14.7 to 14.94 GHz and  $N \sim 30$  in the frequency range 10-10.24 GHz. The wave is localized in the lower and diffusive in the upper frequency range. The sample tube is rotated and vibrated after each measurement of the full transmission matrix to produce a new realization of the random sample. Measurements of spectra of the transmission matrix are made for sample lengths  $L = 23, 40, 61$  and 102 cm. The impact of absorption on the statistics of transmission is removed by Fourier transforming the field spectrum into the time domain and multiplying the time signal by  $\exp^{t/2\tau_a}$ , where  $t$  is the time delay and  $\tau_a$  is the absorption rate [22].

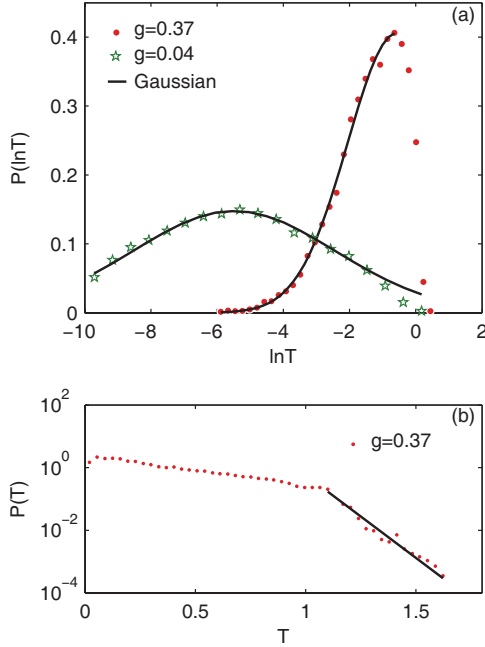


FIG. 1. (Color online) Probability distribution of optical conductance for two random ensembles with values of  $g = 0.37$  and  $0.04$ , respectively. (a)  $P(\ln T)$  for  $g = 0.37$  (red dots) and  $g = 0.04$  (green asterisk). The solid black line is a Gaussian fit to the data, where for  $g = 0.04$ , all the data points are included and for  $g = 0.37$ , only the data to the left of the peak are used in the fit. (b)  $P(T)$  for  $g=0.37$  in a semi-log plot with an exponentially decaying tail.

Measurements of the probability distribution of  $\ln T$ ,  $P(\ln T)$ , for random ensembles with values of the dimensionless conductance of  $g = 0.37$  and  $0.04$  are presented in Fig. 1a.  $P(\ln T)$  is seen to be Gaussian for  $g = 0.04$  and highly asymmetrical for  $g = 0.37$ . The low transmission side of  $P(\ln T)$  is well fit by a Gaussian distribution, and to fall sharply above the peak. For  $T > 1.1$ ,  $P(T)$  is seen in Fig. 1b to fall exponentially in accord with ref. [20].

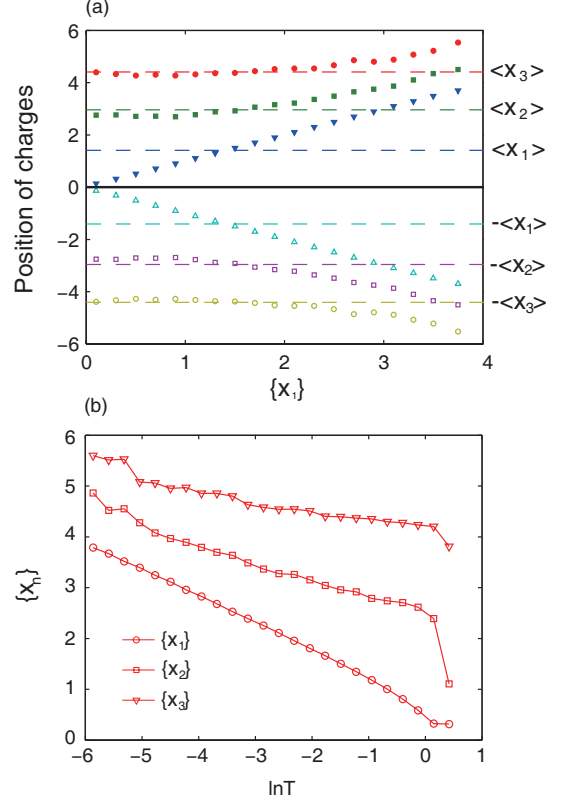


FIG. 2. (Color online) Charge model of transmission eigenvalues and conductance. (a) Average positions of charges and their images with respect to different positions of the first charge  $x_1$  in the random ensemble with  $g = 0.37$ . The dashed lines show the average positions of the charges for this ensemble. (b) Average positions of charges vs.  $\ln T$  in the same ensemble. The curly brackets indicates the averaging is over subset of transmission matrices with the specified value of  $\ln T$ .

These results can be understood with the aid of the charge model in which transmission eigenvalues  $\tau_n$  are associated with positions of parallel line charges at  $x_n$  and their images at  $-x_n$  embedded in a compensating continuous charge distribution [17]. The transmission eigenvalues are expressed as,  $\tau_n = 1/\cosh^2(x_n)$ . The repulsion between two parallel lines of charges with same sign with potential  $\ln|x_i - x_j|$  mimics the interaction between eigenvalues of the random matrix, while the oppositely charged jellium background provides an overall attractive potential that holds the entire structure together.

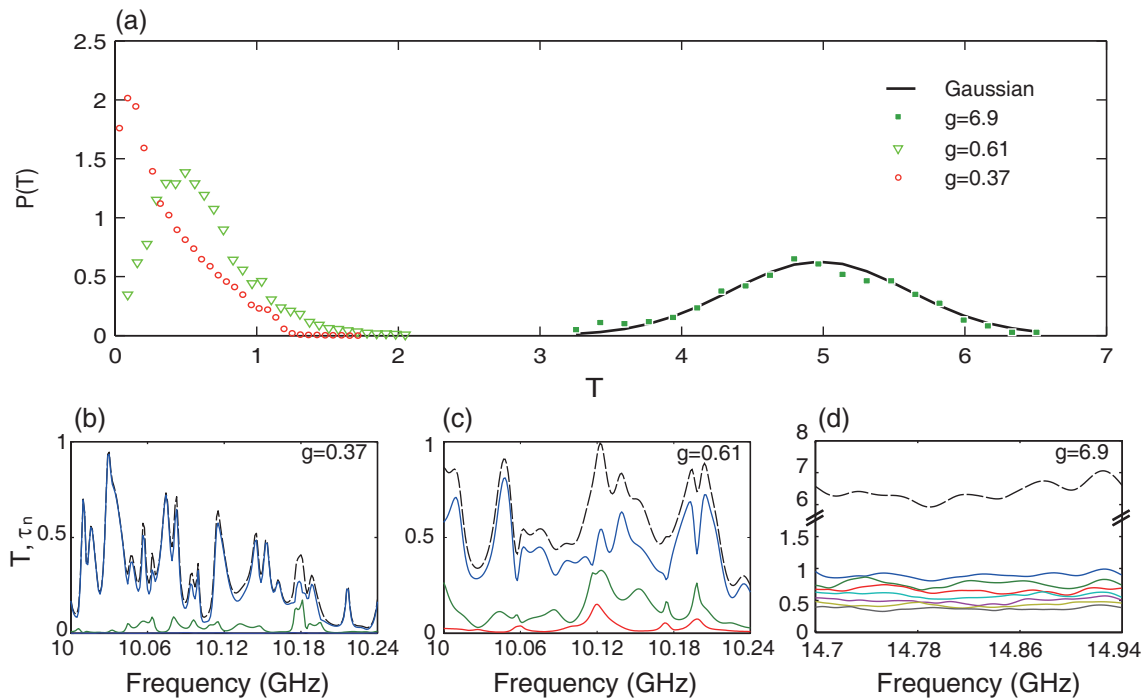


FIG. 3. (Color online) Changing of distribution of optical conductance with respect to different values of  $g$  and the underlying transmission eigenvalues. (a)  $P(T)$  for three values of  $g = 6.9, 0.61$  and  $0.37$ . A Gaussian fit to  $P(T)$  for diffusive waves with  $g = 6.9$  is shown. (b-d) A typical spectra of transmission eigenvalues  $\tau_n$  and  $T$  for (b)  $g = 0.37$ , (c)  $g = 0.61$  and (d)  $g = 6.9$ .

The average spacing between neighboring lines of charge is  $L/\xi$  while the background charge provides a screening length of  $\sqrt{L/\xi}$ , where  $L$  and  $\xi$  are the sample and localization lengths, respectively. The spacing between charges is smaller than the screening length for diffusive waves, for which  $L < \xi$ , so that the Coulomb interaction is not effectively screened and the charges experience a logarithmic repulsion. This is the origin of universal conductance fluctuations for diffusive waves [11]. The repulsion between the first charge  $x_1$  associated with the highest transmission eigenvalues  $\tau_1$  and its image placed at  $-x_1$  naturally provides a ceiling for  $\tau_1$  of unity.

In Fig. 2a, we plot the variation of the average positions of the charges for different values of the position of the first charge  $x_1$  in the random ensemble with  $g = 0.37$ . The average spacing between  $x_1$  and  $x_2$  increases as the value of  $x_1$  decreases. At the same time, the spacing between  $x_2$  and  $x_3$  hardly changes as  $x_1$  moves towards the origin and their average position is little changed. This reflects the tendency to heal large fluctuations in charge positions for more remote charges and is a manifestation of electrical screening in this model.

The source of the sharp cutoff in  $P(\ln T)$  can be seen in the context of the charge model by examining the spacing of charges for different values of  $\ln T$  shown in Fig. 2b together with the average positions of charges for different values of  $x_1$ , shown in Fig. 2a. A relatively high value

of  $T$  can only be obtained when the first charge is near the origin. This is an unlikely circumstance because this charge is strongly repelled by its image.  $P(T)$  would be expected to fall especially rapidly for values of  $T$  above unity since this would require two charges along with their images to be close to the origin.

In Fig. 3, we present  $P(T)$  for  $g = 0.37, 0.61, 6.9$  as well as a spectrum of transmission eigenvalues for a single realization for each ensemble. For diffusive waves with  $g = 6.9$ , many transmission eigenvalues contribute to  $T$  leading to a Gaussian distribution for  $P(T)$ , as seen in Fig. 3a. For  $g \sim 1$ , we have seen that transmission is dominated by the highest transmission eigenvalue,  $\tau_1$  except when  $T$  is comparable or greater than 1.

Deep in the localized regime,  $g \ll 1$ ,  $T$  is essentially given by  $\tau_1$ . In this limit,  $T \sim \tau_1 \sim 4 \exp(-2x_1)$  and  $x_1$  is hypothesized to follow a Gaussian distribution. This leads to the log-normal distribution for  $T$  seen in Fig. 1a for  $g = 0.04$ , as predicted by SPS theory. SPS further predicts that in the limit of small  $g$ ,  $\text{var}(\ln T) \equiv \sigma^2 = -\langle \ln T \rangle$ , so that  $P(\ln T)$  depends only upon the single parameter  $-\langle \ln T \rangle/L = 1/\xi$ . The ratio of  $\sigma^2$  and  $-\langle \ln T \rangle$  vs  $L/\xi$  is plotted in Fig. 4 and seen to approach unity for  $L \gg \xi \sim 24\text{cm}$  in accord with SPS.

In conclusion, we have observed the joint distribution of  $T$  and the transmission eigenvalues  $\tau_n$  in the Anderson localization transition. We have demonstrated that the

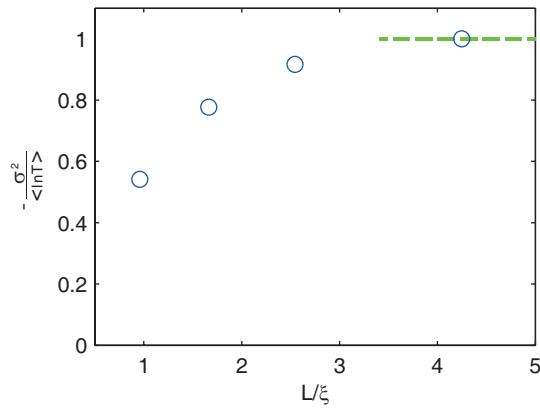


FIG. 4. (Color online) Hypothesis of single parameter scaling theory of localization. The ratio of variance of  $\ln T$  and  $-\langle \ln T \rangle$  vs. the ratio of sample length  $L$  to the localization length  $\xi$ . The dashed green line is the prediction from SPS for  $L \gg \xi$ .

Coulomb charge model including image charges provides an intuitive basis for understanding the statistics of conductance. Since the statistics of transmission for a single incident channel,  $T_{ba}$  and  $T_a$  can be measured for those samples, the full range of statistic of waves of noninteracting particles can be studied in a single system. This demonstrates the power of a unified approach to mesoscopic physics encompassing electronic conductance and statistical optics.

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